

Universal Threshold for the Dynamical Behavior of Lattice Systems with Long-Range Interactions

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Dynamical properties of lattice systems with long-range pair interactions, decaying like $1/r^\alpha$ with the distance r , are investigated, in particular the time scales governing the relaxation to equilibrium. Upon varying the interaction range α , we find evidence for the existence of a threshold at $\alpha = d/2$, dependent on the spatial dimension d , at which the relaxation behavior changes qualitatively and the corresponding scaling exponents switch to a different regime. Based on analytical as well as numerical observations in systems of vastly differing nature, ranging from quantum to classical, from ferromagnetic to antiferromagnetic, and including a variety of lattice structures, we conjecture this threshold and some of its characteristic properties to be universal.

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Most lattice models studied in condensed matter physics have interactions of finite range. Despite the long-range character of the fundamental electromagnetic interactions, the presence of positive and negative charges gives rise to screening effects that cause interactions to be effectively short range and justify an approximation by finite-range (and often even nearest-neighbor) interactions. But a finite-range approximation is not always justified. One obvious example are astrophysical systems dominated by gravitational interactions where screening does not occur. Historically, it was in this context that several of the anomalies of long-range interacting systems were first discussed, for example negative heat capacities of bounded self-gravitating gas spheres [1]. In subsequent decades, a general interest in fundamental issues of statistical physics of long-range interacting systems arose [2].

Technically, interactions of finite range turn out to be convenient: Exact solutions of many-body models, rare as they are, are in most cases restricted to finite-range interactions, and also general theorems in thermostistical physics (like the convexity properties of thermodynamic functions) are often proved under the assumption of short-range interactions [3]. “Short range” here refers not only to interactions of finite range, but also to those decaying like $1/r^\alpha$ with the distance r and an exponent α larger than the spatial dimension d of the system. And indeed, various long-range systems, i.e., those with exponents $\alpha < d$, have been found to violate ensemble equivalence, have negative microcanonical specific heat, and other peculiarities [2].

Out of equilibrium, an intriguing and physically relevant phenomenon that has been observed in long-range systems is the existence of quasistationary states. These are nonequilibrium states whose lifetimes τ diverge with

increasing system size N . As a result, for a sufficiently large system, relaxation to equilibrium takes place on a time scale that is larger than any realistic observation time. Such behavior has been reported for various long-range systems, ranging from classical toy models with mean-field interactions to gravitating systems [4]. The exponent α in these studies is usually kept fixed—either at $\alpha = 0$ (corresponding to mean-field interactions) where analytical calculations are easier, or at some integer number in order to account for classical gravity.

In this Letter we investigate the relaxation to equilibrium of lattice systems with long-range pair interactions, and in particular the α dependence of the relaxation times. We report analytical as well as numerical results on several classes, containing systems of vastly differing nature. For all lattice structures studied, and regardless of whether the systems are quantum or classical, we observe, at a threshold value of $\alpha = d/2$, a drastic change of the relaxational dynamics. In particular, the exponent $q(\alpha)$, governing the scaling $\tau \propto N^{q(\alpha)}$ of the relaxation time τ , switches from one regime to another at $\alpha = d/2$. Certain qualitative aspects of the scaling laws also appear to be universal, as summarized in Figs. 1 and 3. Note that the threshold value $\alpha = d/2$ that is relevant for non-equilibrium phenomena differs from the one at $\alpha = d$ commonly used for distinguishing between long- and short-range behavior in equilibrium statistical mechanics. Other aspects of the dynamics are evidently nonuniversal, even to the point that relaxation in some systems may slow down with increasing system size, but accelerate in others.

This Letter aims at furthering the understanding of fundamental aspects of nonequilibrium statistical mechanics of long-range interacting systems. In particular, the mechanism of relaxation in long-range interacting systems,

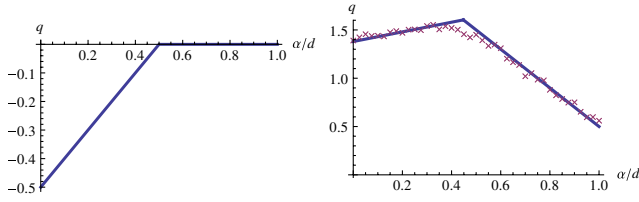


FIG. 1 (color online). The exponent q of the scaling law $\tau \propto N^q$ that governs the system-size dependence of the relaxation time τ . Left: For the long-range quantum Ising model (1) the exponent $q = \min\{0, \alpha/d - 1/2\}$ follows from the bounds in (3). Right: For the αXY chain (4). The crosses mark data points, obtained by a scaling analysis as in Fig. 2. The line is plotted as a guide for the eye, indicating two distinct linear regimes with a crossover at $\alpha/d \approx 0.44$.

but also many aspects of its phenomenology, are still only poorly understood. Identifying universal properties and threshold values may give valuable clues and deepen the general understanding. On the more applied side, recent developments have made long-range interacting systems accessible in the laboratory, and an experimental verification of some of our results should be feasible [5]. Particularly promising for such a check is an ion-trap-based quantum simulator as reported by Britton *et al.* [6]. In this setup, a quantum long-range Ising model is emulated, and the exponent α governing the interaction range can be tuned between 0 and 3.

Long-range quantum Ising model.—The first class of models we are studying consists of spin-1/2 degrees of freedom on the N sites of a lattice Λ , governed by the Hamiltonian operator

$$H_\Lambda = -\frac{J}{2} \sum_{\substack{i,j \in \Lambda \\ i \neq j}} \frac{\sigma_i^z \sigma_j^z}{|i-j|^\alpha} - h \sum_{i \in \Lambda} \sigma_i^z. \quad (1)$$

Here, σ^z is the z component of the Pauli spin operator, h is an external magnetic field, $|i-j|$ denotes the Euclidean distance of sites i and j , and the exponent $\alpha > 0$ determines the interaction range. Depending on the sign of the coupling J , the spin—spin interactions are ferromagnetic ($J > 0$) or antiferromagnetic ($J < 0$). On certain lattices (e.g., a triangular lattice), antiferromagnetic interactions result in geometrical frustration. An important difference to a similar model considered in Ref. [7] is the absence of an N -dependent normalization factor in front of the first sum in (1), introduced in Ref. [7] to ensure extensivity of the energy. We will comment on the role of the factor later in this Letter.

The expectation value $\langle A \rangle$ of an observable A is given by $\text{Tr}[A\rho(t)]$, where the time evolution of the density operator ρ is governed by the von Neumann equation. For all initial density operators ρ_0 that are diagonal in the σ^x tensor product eigenbasis, the time evolution of single-spin observables can be computed analytically in arbitrary spatial dimension, yielding

$$\langle \sigma_i^x \rangle(t) = \langle \sigma_i^x \rangle(0) \cos(2ht) \prod_{\substack{j \in \Lambda \\ j \neq i}} \cos\left(\frac{2Jt}{|i-j|^\alpha}\right) \quad (2)$$

(see Supplemental Material A [8]). For all finite lattices Λ , (2) is a quasiperiodic function. Proper relaxation to equilibrium can be observed in the thermodynamic limit of infinite lattice size, where using techniques analogous to those in Refs. [7,9,10], we obtain

$$|\langle \sigma_i^x \rangle(t)| \leq |\langle \sigma_i^x \rangle(0) \cos(2ht)| \times \begin{cases} \exp\left[-\frac{2^{5+2\alpha-d} \pi^{d/2-2}}{(d-2\alpha)\Gamma(d/2)} J^2 t^2 N^{1-2\alpha/d}\right] & \text{for } 0 \leq \alpha < d/2, \\ \exp\left[-\frac{2^{1+d-2\alpha} \pi^{d/2}}{(2\alpha-d)\Gamma(d/2)} \left|\frac{4Jt}{\pi}\right|^{d/\alpha}\right] & \text{for } \alpha > d/2, \end{cases} \quad (3)$$

valid for large N and t (see Supplemental Material B [8]). This result differs from the one reported in Ref. [7] in a nontrivial way. This is a consequence of the absence of the above-mentioned N -dependent normalization in the Hamiltonian (1), which essentially corresponds to taking the limits of large N and t in a different way. The functional form of the upper bounds in (3) indeed correctly reflects the behavior of $|\langle \sigma_i^x \rangle(t)|$, i.e., the powers of N and t in the exponent agree excellently with a numerical evaluation of (2) for large N , and only the numerical constants are, as expected, overestimated (Supplemental Material B [8]).

From Eq. (3) it follows that a change of regime takes place at $\alpha = d/2$. For $\alpha < d/2$, relaxation to equilibrium is Gaussian in time, and the corresponding relaxation time τ scales like $\tau \propto N^{\alpha/d-1/2}$, shrinking to zero in the limit of large N . For $\alpha > d/2$, relaxation is governed by a

compressed or stretched exponential in t , with a relaxation time that is constant asymptotically for large N . These scaling laws are summarized in Fig. 1 (left). The threshold at $\alpha = d/2$ suggests the following interpretation: Only for $\alpha < d/2$ are the pair interactions sufficiently long-range such that the relaxation dynamics of a single spin is directly influenced, and thereby sped up, by the presence of $\mathcal{O}(N)$ spins. Analytical calculations of spin—spin correlation functions indicate that further significant qualitative and quantitative changes occur in other dynamical quantities [10]. Interestingly, this dynamical long-range threshold differs from the equilibrium threshold at $\alpha = d$ below which peculiar long-range behavior, like nonequivalence of ensembles or negative specific heat, may occur in equilibrium statistical physics.

αXY chain.—This model, introduced in Ref. [11], consists of classical XY spins attached to the sites i of a

one-dimensional chain and parametrized by the angular variables ϕ_i . The time evolution is generated, via Hamilton's equations, by the Hamiltonian function

$$H(p, \phi) = \sum_{i=1}^N \frac{p_i^2}{2} - \frac{J}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{\cos(\phi_i - \phi_j)}{|i - j|^\alpha}, \quad (4)$$

where $\phi = (\phi_1, \dots, \phi_N)$ is the vector of angle variables and $p = (p_1, \dots, p_N)$ the vector of conjugate momenta. For $\alpha = 0$, and besides a normalization factor $1/N$ in front of the second sum, Eq. (4) reduces to the much studied Hamiltonian Mean-Field model [4], a model known to display many of the peculiarities of long-range interacting systems. In particular, relaxation to equilibrium has been studied extensively, and the occurrence of quasistationary states was observed for large classes of initial conditions [12]. In equilibrium and for exponents $0 \leq \alpha < 1$, the model (4) shows a transition from a magnetized phase at energy densities $e = E/N < J/4$ to an unmagnetized phase for $e > J/4$ [13].

Initially we prepare the system in so-called waterbag initial distributions, with initial angles ϕ_i drawn from a flat distribution, and initial momenta taking random values in the symmetric interval $[-\Delta, +\Delta]$ with some $\Delta > 0$; the average energy per particle is then $e = \Delta^2/6$. The time evolution is investigated by numerically integrating the Hamiltonian equations of motion, using a sixth-order symplectic integrator [14]. In earlier studies of the $\alpha = 0$ case, the magnetization m had been monitored over time in order to observe relaxation to equilibrium [15]. This approach is not viable above the critical energy $e_c = 1/4$ where the initial and the equilibrium value of m are both close to zero. For that reason, we monitor the time evolution of the kurtosis of the momentum distribution, $\kappa = \langle p^4 \rangle / \langle p^2 \rangle^2$, where the angular brackets denote averages over the lattice. The kurtosis of the waterbag initial states we are using is $\kappa_0 = 9/5$, whereas the Boltzmann equilibrium distribution has $\kappa_{\text{eq}} = 3$.

Choosing $\Delta = 5/4$, we prepare the system at an energy density $e = 25/96 \approx 0.26$ slightly above the transition energy e_c in the unmagnetized regime. The time evolution of the kurtosis κ is shown in Fig. 2 (left) for $\alpha = 1/8$. The data reveal a relaxation towards the equilibrium value $\kappa_{\text{eq}} = 3$, on a time scale that depends strongly on the system size N . Plotting the same data vs rescaled time t/N^q with $q = 1.453$, the curves for different N collapse onto each other, demonstrating the validity of the scaling law; see Fig. 2 (right). Performing such a scaling analysis for different values of α , we obtain the scaling exponent q as a function of α as shown in Fig. 1 (right). The plot reveals two regimes: For $0 \leq \alpha \leq \alpha_{\text{th}}$ with $\alpha_{\text{th}} \approx 0.45$, q evolves linearly in α as $q \approx 1.38 + 0.5\alpha$. The second regime, again linear in α , is described by $q \approx 2.5 - 2\alpha$ for $\alpha_{\text{th}} \leq \alpha < 1$. This confirms, similar to our findings for the quantum Ising model, the presence of two distinct

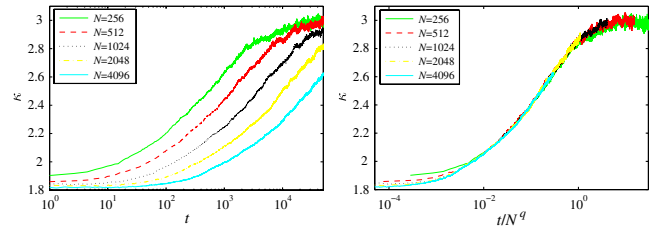


FIG. 2 (color online). The kurtosis κ of the momentum distribution of the αXY chain, plotted for $\alpha = 1/8$, energy density $e \approx 0.26$, and system sizes $N = 256, 512, 1024, 2048$, and 4096 . Data are averaged over 128, 64, 32, 16, and 8 realizations, respectively. Left: As a function of time t , the relaxation time increases with system size N . Right: As a function of rescaled time t/N^q with scaling exponent $q = 1.453$.

power-law regimes for the relaxation times of the αXY chain. For $\alpha > 1$, rescaling of time does not any longer lead to a data collapse, so either no such scaling law exists in this regime, or larger system sizes would be necessary to reach the scaling regime. For other values of the energy density e , the data collapse is of lesser quality, but the behavior of $q(\alpha)$ is similar to the one shown in Fig. 1, though with larger fluctuations (see Supplemental Material C [8] for simulation data and a more detailed discussion).

Normalization of the energy scale.—In many papers on long-range interacting systems, the pair-interaction term in the Hamiltonian is made extensive by equipping it with an N -dependent normalization factor \mathcal{N} . For the models studied in this Letter, such normalization factors, as discussed for example in Refs. [7,11,13], behave as $\mathcal{N} \sim N^{\alpha/d-1}$ asymptotically for large N when $\alpha < d$ (while no normalization is required for $\alpha > d$). Extensivity of the energy is a prerequisite for a well-defined thermodynamic limit when making the transition from equilibrium statistical mechanics to thermodynamics [3]. Moreover, interesting physical phenomena, like equilibrium phase transitions, are caused by competition between energetic and entropic effects, and extensivity of both these quantities is usually required in order for such a competition to survive in the thermodynamic limit.

For studying the approach to equilibrium, a normalization factor is not necessary, but can be included [16]. An N -dependent prefactor in the energy induces a further N dependence of the time scale, shifting the scaling exponent q of the relaxation time scale to $\tilde{q} = q + (1 - \alpha/d)$ in quantum dynamics, and to $\tilde{q} = q + (1 - \alpha/d)/2$ in classical Hamiltonian dynamics. The dependence of the exponent \tilde{q} on α is shown in Fig. 3 for the long-range quantum Ising model and the classical αXY chain. The similarities between the two models become even more evident in this plot, with a crossover from a constant regime for $\alpha < \alpha_{\text{th}}$ to a linear regime above the threshold.

In the case of the long-range quantum Ising model, introducing the normalization factor \mathcal{N} has the effect of even

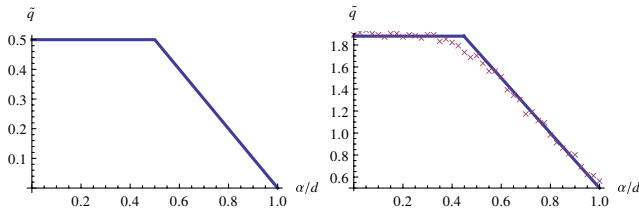


FIG. 3 (color online). As in Fig. 1, but for scaling exponents \tilde{q} as modified by the presence of an N -dependent prefactor \mathcal{N} in the Hamiltonian. The left plot is for the long-range quantum Ising model, the right for the classical αXY chain.

flipping the sign of the exponent, from a negative q to a positive \tilde{q} . This implies that, while the relaxation to equilibrium takes longer and longer with increasing system size in the presence of \mathcal{N} , the opposite happens for the original Hamiltonian: Relaxation speeds up with increasing N , leading to “instantaneous” equilibration in the thermodynamic limit. This is different from what is observed for the αXY chain, where long-lived quasi-stationary states occur independently of whether the factor \mathcal{N} is present or not.

Discussion of the results.—As illustrated in Figs. 1 and 3, we find that the exponent q in the scaling law $\tau \propto N^q$ of the relaxation time varies linearly as a function of α , with a change from one linear regime to another at $\alpha = \alpha_{\text{th}}$. For the quantum Ising model, $\alpha_{\text{th}} = d/2$ is the exact location of this change, whereas for the αXY chain we find an approximate value of $\alpha_{\text{th}} \approx 0.45$. However, results on the largest Lyapunov exponent of the αXY model on d -dimensional lattices [17] support the conjecture that the exact value of α_{th} is $d/2$ also in the classical case: Lyapunov exponents are characteristic quantities for the time evolution, quantifying in some sense the chaoticity of the dynamics. The authors of Ref. [17] find that the largest Lyapunov exponent vanishes like $N^{-\kappa}$, where κ changes from a constant regime for $0 < \alpha/d < 1/2$ to a linearly decreasing regime for $1/2 < \alpha/d < 1$. The qualitative similarity of $\kappa(\alpha)$ in Fig. 3 of Ref. [17] to the plot of $\tilde{q}(\alpha)$ in Fig. 3 of the present Letter is striking. At least sufficiently far from equilibrium, the sum of positive Lyapunov exponents is known to correspond to an entropy rate, which in turn provides a link to the speed at which equilibrium is approached [18]. When the largest term of that sum switches to a different scaling regime, this change is expected to reflect also in the sum and, therefore, in the speed at which the system relaxes to equilibrium. These arguments suggest that the transition from one linear regime of $q(\alpha)$ to another takes place precisely at $\alpha_{\text{th}} = d/2$, not only for the quantum Ising model, but also for the αXY chain, and our numerical results are in good agreement with this analytical prediction.

Our understanding of the origin of the universality of the threshold at $\alpha_{\text{th}} = d/2$ is partial at best, but some physical intuition can be gained from studying Lieb-Robinson-type bounds on the propagation of perturbations. Hastings and

Koma [19] report such a bound for a broad class of quantum lattice systems with long-range interactions. One restriction on the interactions (Eq. (2.3) of Ref. [19]) is essential for obtaining a nontrivial bound, and it roughly amounts to requiring that

$$\sum_{k \in \Lambda} \frac{1}{|i-k|^\alpha} \frac{1}{|j-k|^\alpha} < \infty \quad (5)$$

for any given lattice sites $i, j \in \Lambda$. By integral approximation one finds that (5) is satisfied for $\alpha > d/2$, reproducing the threshold value we found for the relaxation dynamics. This suggests an appealing, though speculative, intuitive explanation: In the regime $\alpha > \alpha_{\text{th}}$, restrictions on the speed at which perturbations propagate (as given by the Lieb-Robinson bound) are responsible for one type of relaxation behavior, whereas the absence of such restrictions gives rise to another type of relaxation in the regime $\alpha < \alpha_{\text{th}}$. However, to turn this intuition into a proof of the universality of the threshold at α_{th} , important pieces are still missing, one of them being a classical version of the Lieb-Robinson bound for long-range interacting systems.

In summary, substantial analytical as well as numerical evidence has been found for the existence of a threshold at $\alpha_{\text{th}} = d/2$ at which dynamical properties of long-range interacting systems show an abrupt change from one regime to another. This threshold is found for classical Hamiltonian as well as quantum dynamics, on lattices of arbitrary dimension and for various lattice structures, and for ferromagnetic as well as anti-ferromagnetic interactions. Anecdotal evidence that the same threshold value is of significance also in random models is reported in Ref. [20], although here different kinds of randomness may behave differently and the scenario becomes more involved [21]. Furthermore, a change of regime can be observed not only in the relaxation times we have studied in the present letter, but also in other dynamical quantities, be it the above-mentioned largest Lyapunov exponents [17] or the emergence of widely separated time scales of decaying correlations [10]. Beyond the universality of the threshold value $\alpha_{\text{th}} = d/2$, it is tempting to conjecture on the basis of Fig. 3 that also the two linear regimes in $q(\alpha)$ or $\tilde{q}(\alpha)$ may be model-independent and universal. Admittedly, this is speculation, and understanding the origin of the observed universality still poses challenges for future research. An experimental verification for more general quantum Ising models (i.e., in the presence of a transverse magnetic field) should be possible with the ion trap technology of Ref. [6].

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